COMPLEX ANALYSIS
TOPIC XVI: SEQUENCES

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Abstract. We outline the development of sequences in $\mathbb{C}$, starting with open and closed sets, and ending with the statement of the Bolzano-Weierstrauss Theorem for complex numbers. Some propositions are formulated as problems.

1. Topology of $\mathbb{C}$

1.1. Open Sets.

Definition 1. Let $u \in \mathbb{C}$. The *open ball around u of radius r* is

$$B_r(u) = \{z \in \mathbb{C} \mid |z - u| < r\}.$$

Let $U \subset \mathbb{C}$. We say that U is *open* if for every $u \in U$ there exists $r > 0$ such that $B_r(u) \subset U$.

Problem 1. Let $U \subset \mathbb{C}$. Show that U is open if and only if U is the union of a collection of open balls.

Problem 2. Let $U, V \subset \mathbb{R}$ be open sets. Show that $U \cap V$ is an open set.

Definition 2. Let X be a set. A *collection of subsets* of X is a set $\mathcal{C}$ whose members are subsets of X. We will use the following notation.

- $\cup\mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\}$
- $\cap\mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}$

Problem 3. Let $\mathcal{T}$ denote the collection of all open subsets of $\mathbb{C}$.

(a) Show that $\varnothing \in \mathcal{T}$ and $\mathbb{C} \in \mathcal{T}$.
(b) Show that if $\mathcal{C} \subset \mathcal{T}$, then $\cup\mathcal{C} \in \mathcal{T}$.
(c) Show that if $\mathcal{C} \subset \mathcal{T}$ and $\mathcal{C}$ is finite, then $\cap\mathcal{C} \in \mathcal{T}$.

The collection $\mathcal{T}$ is known as the *topology* of $\mathbb{C}$.

Definition 3. Let $z \in \mathbb{C}$. A *neighborhood* of z is a subset of $\mathbb{C}$ which contains an open set which contains z.

Problem 4. Let $z \in \mathbb{C}$ and let $A, B \subset \mathbb{C}$ be neighborhoods of z. Show that $A \cap B$ is a neighborhood of z.

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1.2. **Closed Sets.**

Definition 4. Let $F \subset \mathbb{C}$.

We say that F is *closed* if its complement $\mathbb{C} \smallsetminus F$ is open.

Problem 5. Let $\mathcal{F}$ denote the collection of all closed subsets of $\mathbb{C}$.

(a) Show that $\varnothing \in \mathcal{F}$ and $\mathbb{C} \in \mathcal{F}$.
(b) Show that if $\mathcal{C} \subset \mathcal{F}$, then $\cap\mathcal{C} \in \mathcal{F}$.
(c) Show that if $\mathcal{C} \subset \mathcal{F}$ and $\mathcal{C}$ is finite, then $\cup\mathcal{C} \in \mathcal{F}$.

Definition 5. Let $A \subset \mathbb{C}$.

We say that A is *bounded* if there exists $M \in \mathbb{R}$ such that $A \subset B_M(0)$.

We say that A is *compact* if it is closed and bounded.

The word compact has a more general definition in a more general settings, but in the larger sense, a subset of $\mathbb{C}$ is compact if and only if it is closed and bounded. Hence, for our purposes, we simply take it as a definition.

1.3. **Classification of Points.**

Definition 6. Let $A \subset \mathbb{C}$.

An *interior point* of A is a point $z \in A$ such that A contains a neighborhood of z. The *interior* of A is the set of interior points of A and is denoted A°.

Proposition 1. *Let $A \subset \mathbb{C}$. Then:*

(a) *A is open if and only if $A = A^\circ$;*
(b) *A is open if and only if every point in A is an interior point;*
(c) *The interior of A is the union of all open sets which are contained in A.*

Definition 7. Let $A \subset \mathbb{C}$. A *closure point* of A is a point $z \in \mathbb{C}$ such that every neighborhood of z intersects A. The *closure* of $A \subset X$ is the set of closure points of A and is denoted $\overline{A}$.

Proposition 2. *Let $A \subset \mathbb{C}$. Then:*

(a) *A is closed if and only if $A = \overline{A}$;*
(b) *A is closed if and only if every point in A is an closure point;*
(c) *Then $\overline{A}$ is the intersection of the closed subsets of $\mathbb{C}$ which contain A.*

Definition 8. Let $A \subset \mathbb{C}$. A *boundary point* of A is a point $z \in \mathbb{C}$ such that every neighborhood of z intersects A and A^c. The *boundary* of A is the set of boundary points of A and is denoted ∂A.

Proposition 3. *Let $A \subset \mathbb{C}$. Then*

(a) $\partial A = \overline{A} \smallsetminus A^\circ$;
(b) $\partial A = \overline{A} \cap \overline{A^c}$;
(c) $\partial A = \partial A^c$;
(d) $\overline{A} = A \cap \partial A$;
(e) $A^\circ = A \smallsetminus \partial A$;
(f) $\partial(\partial A) \subset \partial A$;
(g) $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$.

Definition 9. Let $A \subset \mathbb{C}$. An *accumulation point* of A is a point $z \in \mathbb{C}$ such that every deleted neighborhood of z intersects A. The *derived set* of A is the set of accumulation points of A and is denoted A'.

2. Sequences

Definition 10. Let X be a set. A *sequence* in X is a function $a : \mathbb{N} \to X$.

We may write a_n to mean $a(n)$. Next we specify notation to indicate the entire sequence as opposed to a specific member of the range.

We often think of a sequence as an infinitely long tuple of elements from X, so it looks like $(a_1, a_2, a_3, \dots)$. This is written more succinctly as $(a_n)_{n=1}^{\infty}$, or $(a_n)_{n\in\mathbb{N}}$, or simply (a_n).

In the case $X \subset \mathbb{C}$, we call this a sequence of complex number numbers. We are also interested in sequences of real numbers, which become an important special case.

Example 1. We give some famous examples of sequences of real numbers.

- The natural number: $1, 2, 3, 4, \dots, n, \dots.$
- Even numbers: $2, 4, 6, 8, 10, 12, \dots.$
- Powers of two: $2, 4, 8, 16, 32, 64, \dots.$
- The prime numbers: $2, 3, 5, 7, 11, 13, \dots.$
- The square numbers: $1, 4, 9, 16, 25, \dots, n^2, \dots.$
- The triangular numbers: $1, 3, 6, 10, 15, \dots, \frac{n(n+1)}{2}, \dots.$
- An alternating sequence: $1, -1, 1, -1, \dots, (-1)^{n+1}, \dots.$
- The harmonic sequence: $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots.$
- The Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, 21, \dots.$
- The digits of pi: $3, 1, 4, 1, 5, 9, \dots.$

The Recursion Theorem tells us that inductively defined sequences exist. That is, if we have a function $f : X \to X$ and we pick any point in X, and we follows where f sends it as f is repeatedly applied, we can capture the entire journey of the point with a single function.

Theorem 1. (Recursion Theorem)
Let X be a set, $f : X \to X$, and $x_0 \in X$. Then there exists a unique function $x : \mathbb{N} \to X$ such that $x(0) = x_0$ and $x(n+1) = f(x(n))$ for all $n \in \mathbb{N}$.

Example 2. Let $X = \mathbb{R}$ and let $f(x) = 2x - 1$. Set $x_0 = 1$. Then the recursively defined sequence given by $x_{n+1} = f(x_n)$ produces the sequence $1, 3, 5, 7, \dots.$

Example 3. It is possible that x_{n+1} may depend on any previous terms. The Fibonacci sequence is defined in this way as follows.

Let $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_n + F_{n+1}$. This produces the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots.$

Example 4. Let $X = \mathbb{C}_\infty$ be the Riemann sphere, and let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $T(z) = \frac{az+b}{cz+d}$ be a Möbius transformation. Let $z_0 \in \mathbb{C}_\infty$ and set $z_{n+1} = T(z_n)$. Then (z_n) is a recursively defined sequence.

We visualize that the (z_n) is a sequence of points on the Riemann sphere. We ask if this sequence of points gets approaches a specific point, or if it just kind of purposelessly wanders around the sphere forever. We need the concept of limits to make this question precise.

3. Limits of Sequences

Definition 11. Let (a_n) be a sequence of complex numbers and let $p \in \mathbb{C}$. We say that the sequence *converges* to p if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |a_n - p| < \epsilon.$$

If (a_n) converges to p, we call p the *limit* of the sequence, and write $\lim a_n = p$.

The next definition, and the following solved problem, will give us additional means to visualize limits.

Definition 12. Let (a_n) be a sequence in a set X.

We say that (a_n) is *injective* if $a_m = a_n \Rightarrow m = n$.

The *image* of (a_n) is

$$\{a_n\} = \{x \in X \mid x = a_n \text{ for some } n \in \mathbb{N}\}.$$

The N^{th} *tail* of (a_n) is

$$\{a_n : N\} = \{x \in X \mid x = a_n \text{ for some } n \geq N\}.$$

Problem 6. Let (a_n) be a sequence of complex numbers and let $p \in \mathbb{C}$.
Show that the following conditions are equivalent:
(L1) For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - p| < \epsilon$.
(L2) For every neighborhood U of p there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$.
(L3) Every neighborhood of p contains a tail of (a_n).
(L4) Every neighborhood of p contains a_n for all but finitely many $n \in \mathbb{N}$.

Solution.

(**L1** $\Rightarrow$ **L2**) Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - p| < \epsilon$. Let U be a neighborhood of p. Then there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset U$. Let N be so large that $|a_n - p| < \epsilon$ whenever $n \geq N$. Then for $n \geq N$, we have $a_n \in B_\epsilon(p) \subset U$.

(**L2** $\Rightarrow$ **L3**) Suppose that for every neighborhood U of p there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$. Let U be a neighborhood of p and let N be so large that $n \geq N \Rightarrow a_n \in U$. Then $\{a_n \mid n \geq N\} \subset U$, so U contains the N^{th} tail of (a_n).

(**L3** $\Rightarrow$ **L4**) Suppose that every neighborhood U of p contains a tail of (a_n). Let U be a neighborhood of p and let $N \in \mathbb{N}$ such that $\{a_n \mid n \geq N\} \subset U$. If $a_n \notin U$ for some $n \in \mathbb{N}$, then $a_n \notin \{a_n \mid n \geq N\}$, so $n < N$. There are only finitely many such n.

(**L4** $\Rightarrow$ **L1**) Suppose that every neighborhood of p contains a_n for all but finitely many n. Let $\epsilon > 0$. Then $B_\epsilon(p)$ is a neighborhood of p, so $a_n \in B_\epsilon(p)$ for all but finitely many $n \in \mathbb{N}$. The maximum of a finite set of natural numbers always exists. Let $N = 1 + \max\{n \in \mathbb{N} \mid a_n \notin B_\epsilon(p)\}$. Then for $n > N$, we have $|a_n - p| < \epsilon$. $\square$

Problem 7. Let (a_n) be an injective sequence and let $p \in \mathbb{C}$. Show that (a_n) converges to p if and only if every neighborhood of p contains a for all but finitely many $a \in \{a_n\}$.

Problem 8. Find an example of a noninjective sequence (a_n) of real numbers, and a real number p, such that every neighborhood of p contains all but finitely many points in $\{a_n\}$, but (a_n) does not converge to p.

4. Cluster Points of Sequences

Definition 13. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{C}$.
We say that the sequence *clusters* at q if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \ni |a_n - q| < \epsilon.$$

If (a_n) clusters at q, we call q a *cluster point* of (a_n).

Problem 9. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{C}$.
Show that the following conditions are equivalent:
(C1) For every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - q| < \epsilon$.
(C2) For every neighborhood U of q and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $a_n \in U$.
(C3) Every neighborhood of q intersects every tail of (a_n).
(C4) Every neighborhood of q contains a_n for infinitely many $n \in \mathbb{N}$.

Solution.
($\mathbf{C1} \Rightarrow \mathbf{C2}$) Suppose that for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - q| < \epsilon$. Let U be a neighborhood of q and let $N \in \mathbb{N}$. Then there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset U$; thus there exists $n \geq N$ such that $|a_n - q| < \epsilon$. But this says that $a_n \in B_\epsilon(q)$, so $a_n \in U$.

($\mathbf{C2} \Rightarrow \mathbf{C3}$) Suppose that for every neighborhood U of q and every $N \in \mathbb{N}$ there exists $n > N$ such that $a_n \in U$. Let U be a neighborhood of q and let $\{a_n \mid n \geq N\}$ be an arbitrary tail of (a_n). Then for some $n \geq N$, we have $a_n \in U$. But $a_n \in \{a_n \mid n \geq N\}$, so $a_n \in \{a_n \mid n \geq N\} \cap U$, and $\{a_n \mid n \geq N\}$ intersects U.

($\mathbf{C3} \Rightarrow \mathbf{C4}$) Suppose that every neighborhood of q intersects every tail of (a_n). Let U be a neighborhood of q. Suppose by way of contradiction that U contains a_n for only finitely many $n \in \mathbb{N}$. Let m be the largest natural number such that $a_m \in U$. Then $\{a_n : m + 1\}$ is a tail of (a_n) which does not intersect U; this is a contradiction.

($\mathbf{C4} \Rightarrow \mathbf{C1}$) Suppose that every neighborhood of q contains a_n for infinitely many $n \in \mathbb{N}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Then $U = B_\epsilon(q)$ is a neighborhood of q, and U contains a_n for infinitely many $n \in \mathbb{N}$. One such n must be larger than N; if $n \in \mathbb{N}$ such that $a_n \in U$, then $|a_n - q| < \epsilon$. □

Problem 10. Let (a_n) be a sequence of complex numbers and let $p \in \mathbb{C}$. Show that if (a_n) converges to p, then (a_n) clusters at p, and p is the only cluster point.

Solution. Suppose that (a_n) converges to p. Then every neighborhood of p contains a_n for all but finitely many n. Thus there are infinitely many n such that a_n is in the neighborhood. By Problem 9 **(d)**, (a_n) clusters at p.

To see that p is the only cluster point, let $q \in X$, $q \neq p$; we show that (a_n) does not cluster at q. Let $\epsilon = \frac{|p-q|}{2}$ and let $U = B_\epsilon(p)$ and $V = B_\epsilon(q)$. Then U and V are disjoint neighborhoods of p and q respectively.

Let A be a tail of (a_n) such that $A \subset U$. Since $U \cap V = \varnothing$, we have $A \cap V = \varnothing$, so V is a neighborhood of q which does not intersect A. Thus (a_n) does not cluster at q, by 9 **(c)**. □

Problem 11. Find an example of a sequence (a_n) of real numbers and a real number $q \in \mathbb{C}$ such that (a_n) clusters at q but does not converge to q.

Problem 12. Let (a_n) be an injective sequence and let $q \in \mathbb{C}$. Show that (a_n) clusters at q if and only if every neighborhood of q contains a for infinitely many $a \in \{a_n\}$.

Problem 13. Find an example of a noninjective sequence (a_n) of real numbers, a real number q, and a neighborhood U of q, such that (a_n) clusters at q but U contains only finitely many points from $\{a_n\}$.

Problem 14. Let (a_n) be a bounded sequence of real numbers, and set

$$C = \{q \in \mathbb{C} \mid q \text{ is a cluster point of } (a_n)\}.$$

Show that C is closed and bounded.

Solution. First we show that C is closed. Let $w \in C^c$. Then w is not a cluster point of (a_n), so there exists an open neighborhood U of w which does not intersect $\{a_n\}$. If $u \in U$, then U is an open neighborhood of u which does not intersection $\{a_n\}$, so u is not a cluster point of (a_n), so $u \in C^c$. Thus $U \subset C^c$, which shows that C^c is open, since every point in C^c is an interior point. Thus C is closed.

Next we show that C is bounded. Since (a_n) is bounded, there exists $M > 0$ such that $|a_n| < M$ for all $n \in \mathbb{N}$. Suppose that $q \in \mathbb{C}$ such that $|q| > M$. Let $\epsilon = |q| - M$. For $x \in B_\epsilon(q)$, $|x| > M$, so $x \notin \{a_n\}$, so x is not a cluster point of (a_n). This shows that if $q \in C$, then $|q| \leq M$, so C is bounded. □

5. Bounded and Monotone Sequences

Definition 14. A sequence (a_n) of complex numbers is *bounded* if there exists a $R > 0$ such that $|a_n| < R$ for all $n \in \mathbb{N}$.

Problem 15. Let (a_n) be a convergent sequence of complex numbers. Then (a_n) is bounded.

Definition 15. A sequence (a_n) of real numbers is *increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$, and it is *decreasing* if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$. It is *monotone* if it is either increasing of decreasing.

Proposition 4. (Bounded Monotone Convergence Rule)
A bounded monotone sequence of real numbers converges.

Reason. This may be proven using the Dedekind completeness axiom of the real numbers, which says that every set of real numbers which is bounded above has a least upper bound. □

Problem 16. (Squeeze Law)
Let (a_n), (b_n), and (c_n) be sequences in $\mathbb{R}$. Suppose that $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$. Show that if (a_n) and (b_n) both converge to $L \in \mathbb{R}$, then (c_n) also converges to L.

6. Arithmetic of Sequences

Problem 17. Let (a_n) and (b_n) be sequences in $\mathbb{C}$. Suppose that $\lim a_n = L$ and $\lim b_n = M$ for some $L, M \in \mathbb{C}$. Apply the definition to show that the sequence $(a_n + b_n)$ converges to $L + M$.

Solution. Let $\epsilon > 0$.

Let N_1 be so large that $n \geq N_1$ implies $|a_n - L| < \epsilon/2$.

Let N_2 be so large that $n \geq N_2$ implies $|b_n - M| < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$.

Then, for $n \geq N$, we have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Problem 18. Let (a_n) be a sequences in $\mathbb{C}$ and let $c \in \mathbb{C}$. Suppose that $\lim a_n = L$ for some $L \in \mathbb{C}$. Apply the definition to show that the sequence (ca_n) converges to cL.

Problem 19. Let (a_n) and (b_n) be sequences in $\mathbb{C}$. Suppose that $\lim a_n = L$ and $\lim b_n = M$ for some $L, M \in \mathbb{C}$. Apply the definition to show that the sequence $(a_n \cdot b_n)$ converges to LM.

Solution. Let $\epsilon > 0$.

Since (b_n) converges, it is bounded; let $B > 0$ be so large that $|b_n| < B$ for all $n \in \mathbb{N}$.

Let N_1 be so large that $n \geq N_1$ implies $|a_n - L| < \dfrac{\epsilon}{2B}$.

Let N_2 be so large that $n \geq N_2$ implies $|b_n - M| < \dfrac{\epsilon}{2|L|}$.

Let $N = \max\{N_1, N_2\}$.

Then, for $n \geq N$, we have

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - Lb_n + Lb_n - LM| \\ &\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\ &= |a_n - L||b_n| + |L||b_n - M| \\ &< \frac{\epsilon}{2B} \cdot B + |L|\frac{\epsilon}{2|L|} \\ &= \epsilon. \end{aligned}$$

□

7. Cauchy sequences

Definition 16. Let (a_n) be a sequence of complex numbers. We say that (a_n) is a *Cauchy sequence* if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \ni m, n \geq N \Rightarrow |a_m - a_n| < \epsilon.$$

Proposition 5. *Let (a_n) be a sequence of complex numbers. If (a_n) converges, then (a_n) is a Cauchy sequence.*

Proof. Let $\lim a_n = L$.

Let $\epsilon > 0$, and let N be so large that $n \geq N$ implies that $|a_n - L| < \epsilon$.

Then, for $m, n \geq N$, we have

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Definition 17. A subset of $X \subset \mathbb{C}$ is called *complete* if every Cauchy sequence in X converges to a point in X.

Theorem 2. *The set $\mathbb{C}$ is complete.*

Proof. This follows from the fact that $\mathbb{R}$ is complete, which requires a formal definition of the real numbers. □

8. Subsequences

Definition 18. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence in a set X. A *subsequence* of (a_n) is a sequence $b : \mathbb{N} \to X$ which can be expressed as a composition $b = a \circ n$, where $n : \mathbb{N} \to \mathbb{N}$ is an increasing sequence of natural numbers. For each natural number k, we write n_k instead of $n(k)$; thus $b(k) = a(n(k)) = a(n_k) = a_{n_k}$. Thus we may write (a_{n_k}) is indicate a subsequence of (a_n).

Definition 19. Let (a_n) be a sequence of real numbers. A *subsequential limit* of (a_n) is a real number $q \in \mathbb{C}$ such that there exists a convergent subsequence (a_{n_k}) whose limit is q.

Problem 20. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{C}$. Show that q is a cluster point of (a_n) if and only if q is a subsequential limit of (a_n).

Problem 21 (Bolzano-Weierstrauss Theorem)**.** Show that every bounded sequence of complex numbers has a convergent subsequence.

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